MATH 5061 Problem Set 3[1]
Due date: Mar 3, 2021

Problems: (Please hand in your assignments via Blackboard. **Late submissions will not be accepted.**)

Throughout this assignment, we use (M, g) to denote a smooth n-dimensional Riemannian manifold with its Levi-Civita connection ∇ unless otherwise stated.

1. Prove that the antipodal map $A(p) = -p$ induces an isometry on $\mathbb{S}^n$. Use this to introduce a Riemannian metric on $\mathbb{RP}^n$ such that the projection map $\pi : \mathbb{S}^n \to \mathbb{RP}^n$ is a local isometry.

2. Show that the isometry group of $\mathbb{S}^n$, with the induced metric from $\mathbb{R}^{n+1}$, is the orthogonal group $O(n+1)$.

3. For any smooth curve $c : I \to M$ and $t_0, t \in I$, we denote the parallel transport map as $P = P_{c,t_0,t} : T_{c(t_0)}M \to T_{c(t)}M$ along c from $c(t_0)$ to $c(t)$.

   (a) Show that P is a linear isometry. Moreover, if M is oriented, then P is also orientation-preserving.

   (b) Let X, Y be vector fields on M, $p \in M$. Suppose $c : I \to M$ is an integral curve of X with $c(t_0) = p$. Prove that
   $$(\nabla_X Y)(p) = \left.\frac{d}{dt}\right|_{t=t_0} P^{-1}_{c,t_0,t}(Y(c(t))).$$

4. Let TM be the tangent bundle of M. Let $(p, v) \in TM$, i.e. $v \in T_pM$, and $V, W \in T_{(p,v)}(TM)$. Choose curves $\alpha(t) = (p(t), v(t))$ and $\beta(s) = (q(s), w(s))$ in TM with $p(0) = q(0) = p$, $v(0) = w(0) = v$, and $V = \alpha'(0)$, $W = \beta'(0)$. Define an inner product on TM by
   $$\langle V, W\rangle_{(p,v)} = \langle d\pi(V), d\pi(W)\rangle_p + \langle \frac{Dv}{dt}(0), \frac{Dw}{ds}(0)\rangle_p.$$
   Here, $d\pi$ is the differential of the projection map $\pi : TM \to M$.

   (a) Prove that $\langle\cdot,\cdot\rangle$ is well-defined and defines a Riemannian metric on TM.

   (b) A vector at $(p, v) \in TM$ is called *horizontal* if it is $\langle\cdot,\cdot\rangle$-orthogonal to the fiber $\pi^{-1}(p) \cong T_pM$. A curve $c : I \to TM$ is called *horizontal* if its tangent vector is everywhere horizontal. Prove that a curve $c(t) = (p(t), v(t))$ is horizontal if and only if $v(t)$ is a parallel vector field along the curve $t \mapsto p(t)$ in M.

   (c) Prove that the geodesic field is a horizontal vector field.

   (d) Prove that the trajectories of the geodesic field are geodesics on TM with respect to $\langle\cdot,\cdot\rangle$.

[1] Last revised on February 28, 2021